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# Particle systems with stochastic passing

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We study a system of particles moving on a line in the same direction. Passing is allowed and when a fast particle overtakes a slow particle, it acquires a new velocity drawn from a distribution $P_0(v)$, while the slow particle remains unaffected. We show that the system reaches a steady state if $P_0(v)$ vanishes at its lower cutoff; otherwise, the system evolves indefinitely.

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We study a system of interacting particles moving on the real line in one direction, say to the right. The system is endowed with the following dynamics: (i) Particles move freely between ‘‘collisions’’; (ii) After a collision, or ‘‘passing’’ event, the velocity of the slow particle remains the same, $v_{\rm slow}=$const, while the fast particle instantaneously acquires some new velocity, $v_{\rm new}>v_{\rm slow}$, drawn from the intrinsic velocity distribution $P_0(v)$. We want to answer basic questions about the behavior of the system such as, does the velocity distribution $P(v,t)$ reach a steady state or does the system continue to evolve indefinitely? How does the average velocity depend on time?

Our motivation is primarily conceptual, as we want to understand nonequilibrium infinite particle systems with two-body interactions. Thus, we have chosen the simplest dynamics, interactions occur only upon colliding, and only one particle is affected. The appealing simplicity of the model suggests that it might arise in different natural phenomena, and indeed, we originally arrived at this model in an attempt to mimic traffic on one-lane roads. Somewhat related dynamics were already used in modeling voting systems [1,2], force fluctuations in bead packs [3], asset exchange processes [4], combinatorial processes [5], continuous asymmetric exclusion processes [6], granular gases [7], and aggregation-fragmentation processes [8].

Let us first consider discrete velocity distributions. Specifically, we assume that both initial velocities and new velocities are drawn from the same intrinsic distribution $P_0(v)=\Sigma p_j\delta(v-v_j)$. For the binary distribution, the system does not evolve at all, so the first nontrivial case is the ternary intrinsic distribution when the system contains slow, moderate, and fast particles. Initially,

$$P_0(v)=p_1\delta(v-v_1)+p_2\delta(v-v_2)+p_3\delta(v-v_3). \quad (1)$$

We set

$$p_1+p_2+p_3=1, \quad v_1<v_2<v_3, \quad (2)$$

without a loss of generality. When the steady state is reached, the velocity distribution remains ternary,

$$P_{\rm eq}(v)=p_1\delta(v-v_1)+q_2\delta(v-v_2)+q_3\delta(v-v_3). \quad (3)$$

The density of slow particles does not change, while the densities $q_2$ and $q_3$ of moderate and fast particles differ from the initial values. The final densities are found from a simple probabilistic argument based on the requirement of stationarity. For moderate particles we get

$$p_3(v_2-v_1)q_2=p_2(v_3-v_1)q_3. \quad (4)$$

The left-hand side of Eq. (4) gives the loss in $q_2$, which happens when a moderate particle overtakes a slow particle and becomes a fast particle. The right-hand side gives the gain in $q_2$, which takes place when a fast particle overtakes a slow particle and converts into a moderate particle. Solving Eq. (4) together with the normalization condition, $p_1+q_2+q_3=1$, we find

 

$$q_2=p_2\frac{p_2+p_3}{p_2+\nu p_3},\quad q_3=\nu p_3\frac{p_2+p_3}{p_2+\nu p_3}, \tag{5}$$

where $\nu=(v_2-v_1)/(v_3-v_1)$. Since $\nu<1$, we have $q_2>p_2$ and $q_3<p_3$. Thus, the density of moderate particles increases while the density of fast particles decreases. Similarly, one can analyze discrete velocity distributions with more than three particle species. In all cases, (i) the system reaches a steady state, (ii) the average velocity decreases and eventually reaches some finite value, and (iii) the density of slow particles remains unchanged.

Now we turn to continuous velocity distributions. Let $[v_{\rm min},v_{\rm max}]$ be a support of the intrinsic velocity distribution $P_0(v)$. By Galilean transform, we can set $v_{\rm min}=0$ without loss of generality. We consider unbounded distributions, $v_{\rm max}=\infty$, although main results equally apply to the case of finite $v_{\rm max}$.

The passing rule asserts that when a fast particle overtakes a slow particle moving with a velocity $v_{\rm slow}$, the assignment of the new velocity $v$ occurs with probability

$$P_0(v|v_{\rm slow})=P_0(v)\frac{\theta(v-v_{\rm slow})}{\int_{v_{\rm slow}}^{\infty}dv'P_0(v')}. \tag{6}$$

Equation (6) guarantees that $v>v_{\rm slow}$ and that the normalization requirement, $\int dv\,P_0(v|v_{\rm slow})=1$, is obeyed.

Now we can write a Boltzmann equation for the velocity distribution $P(v,t)$,

$$\begin{aligned}\frac{\partial P(v,t)}{\partial t}=&-P(v,t)\int_0^v dv'(v-v')P(v',t)\\&+\int_0^v dv_2P_0(v|v_2)\int_{v_2}^{\infty}dv_1(v_1-v_2)\\&\times P(v_1,t)P(v_2,t).\end{aligned} \tag{7}$$

The first term on the right-hand side of Eq. (7) describes loss in $P(v,t)$ due to collisions with more slow particles: Collisions occur with a rate proportional to velocity difference, and the integration limits ensure that only collisions with slower particles are taken into account. The second, a gain term, accounts for the increase of $P(v,t)$ due to a random assignment of velocity $v$ after collision.

We could not solve Eq. (7) in the general case of an arbitrary intrinsic velocity distribution $P_0(v)$. Attempts to find a solution even for some particularly simple $P_0(v)$, e.g., linear, exponential, or uniform, turned out to be fruitless as well. Thus, we proceed by employing asymptotic, approximate, and numerical techniques.

We start by looking at the asymptotic behavior of $P(v)$ in the small velocity limit. Let $v\ll u(t)$, where $u(t)$ is the average velocity,

$$u(t)\equiv\langle v\rangle=\int_0^{\infty}dv\,v\,P(v,t). \tag{8}$$

Then Eq. (7) simplifies to

$$\begin{aligned}\frac{\partial P(v,t)}{\partial t}=&P_0(v)u(t)\int_0^v dv_2P(v_2,t)\\&-P(v,t)\int_0^v dv'(v-v')P(v',t).\end{aligned} \tag{9}$$

To probe the small $v$ behavior, we need to know $P_0(v)$ at $v\to0$. Let us consider a family of intrinsic velocity distributions that behave algebraically,

$$P_0(v)\simeq Av^{\mu}\quad\text{when}\quad v\to0. \tag{10}$$

Now *assume* that the system reaches the steady state: $P(v,t)\to P_{\rm eq}(v)$ and $u(t)\to u_{\rm eq}$. Plugging these expressions and Eq. (10) into Eq. (9), we find that $P_{\rm eq}(v)$ also behaves algebraically in the small velocity limit,

$$P_{\rm eq}(v)\simeq(\mu+1)Au_{\rm eq}v^{\mu-1}\quad\text{when}\quad v\to0. \tag{11}$$

By inserting this asymptotics into the normalization condition, $\int_0^\infty dv\,P(v)=1$, we see that it could hold only when $\mu>0$. Thus, our *assumption* that the system reaches a steady state is certainly wrong when $\mu\leq0$. In this region the system will evolve indefinitely. Note that both the exponential and uniform intrinsic distributions belong to the borderline case of $\mu=0$ that separates stationary and evolutionary regimes; for them an anomalously slow kinetics is expected.

To probe the behavior of evolving systems, we *assume* that in the long-time limit there is a very small fraction of ''active'' particles that move with velocities $v\sim1$ and the vast majority of ''creeping'' particles that hardly move at all. We ignore collisions between active particles since their density is very low. We also ignore collisions between creeping particles since their relative velocity is very small. This picture suggests that only collisions between active and creeping particles matter. Hence, the velocity distribution of active particles obeys

$$\frac{\partial P(v,t)}{\partial t}=P_0(v)u(t)-vP(v,t). \tag{12}$$

Equation (12) may at best describe the evolution process in the long-time limit. However, for the sake of tractability, we apply it to the entire time range and use the natural initial condition $P(v,0)=P_0(v)$. Solving Eq. (12) gives

$$P(v,t)=P_0(v)e^{-vt}\left[1+\int_0^t dt'u(t')e^{vt'}\right]. \tag{13}$$

This solution implies $P(v,t)\sim u(t)v^{-1}P_0(v)$ for $v\gg t^{-1}$, which resembles Eq. (11).

To close the solution of Eq. (13), we must determine $u(t)$. It is possible to plug Eq. (13) into the definition of the average velocity, Eq. (8), and get an integral equation for $u(t)$. In the following we use another approach, which is technically simpler. Note that the density of active particles, $\int dv\,P(v,t)$, is manifestly conserved by Eq. (12). After integration over velocity, Eq. (13) becomes

$$1=\hat{P}_0(t)+\int_0^t dt'u(t')\hat{P}_0(t-t'), \tag{14}$$

where $\hat{P}_0(t)=\int_0^\infty dv\,P_0(v)e^{-vt}$ is the Laplace transform of the intrinsic velocity distribution. One can guess the long time behavior of the average velocity without actually solving Eq. (14). Let us assume that the average velocity varies slowly with $t$. Then the integral on the right-hand side of Eq. (14) can be estimated as $u(t)\int_0^t dt'\hat{P}_0(t')$, which implies

$$u(t)\sim\left[\int_0^t dt'\hat{P}_0(t')\right]^{-1}. \quad (15)$$

For an intrinsic velocity distribution with an algebraic behavior (10) in the small-$v$ limit, we have $\hat{P}_0(t)\sim t^{-1-\mu}$ for large $t$. Hence $\int_0^t dt'\hat{P}_0(t')\sim t^{-\mu}$ for $\mu<0$, and it follows from Eq. (15) that $u(t)\sim t^{\mu}$.

The above derivation is careless, though the final asymptotics is correct. To determine $u(t)$ rigorously, we apply the Laplace transform as it is suggested by the convolution form of the integral in Eq. (14). This recasts Eq. (14) into

$$\frac{1}{s}=\int_0^\infty dv\,\frac{P_0(v)}{s+v}+\hat{u}(s)\int_0^\infty dv\,\frac{P_0(v)}{s+v}, \quad (16)$$

where

$$\hat{u}(s)=\int_0^\infty dt\,u(t)e^{-st}. \quad (17)$$

Note that the double Laplace transform of $P_0(v)$ has been simplified by using the identity

$$\int_0^\infty dt e^{-st}\int_0^\infty dv\,P_0(v)e^{-vt}=\int_0^\infty dv\,\frac{P_0(v)}{s+v}. \quad (18)$$

Thus the Laplace transform of the average velocity is

$$\hat{u}(s)=-1+\left[s\int_0^\infty dv\,\frac{P_0(v)}{s+v}\right]^{-1}. \quad (19)$$

Generally, one cannot obtain more explicit results. Given that the above approach describes only the long-time asymptotics, let us focus on this regime. To probe the long-time behavior, one should determine the small $s$ asymptotics of $\hat{u}(s)$. For algebraic intrinsic velocity distributions (10), the asymptotics of Eq. (18) reads

$$\int_0^\infty dv\,\frac{P_0(v)}{s+v}\to As^{\mu}\int_0^\infty dw\,\frac{w^{\mu}}{w+1}=\frac{A\pi}{\sin(-\pi\mu)}s^{\mu}. \quad (20)$$

This applies for $-1<\mu<0$ [the lower bound comes from the normalization requirement, $\int dv\,P_0(v)=1$]. Plugging Eq. (20) into Eq. (19) yields

$$\hat{u}(s)\to\frac{\sin(-\pi\mu)}{A\pi}s^{-1-\mu} \quad \text{for } s\to 0, \quad (21)$$

and by making the inverse Laplace transform, we finally arrive at

$$u(t)\to\frac{\sin(-\pi\mu)}{A\pi\Gamma(1+\mu)}t^{\mu} \quad \text{for } t\to\infty. \quad (22)$$

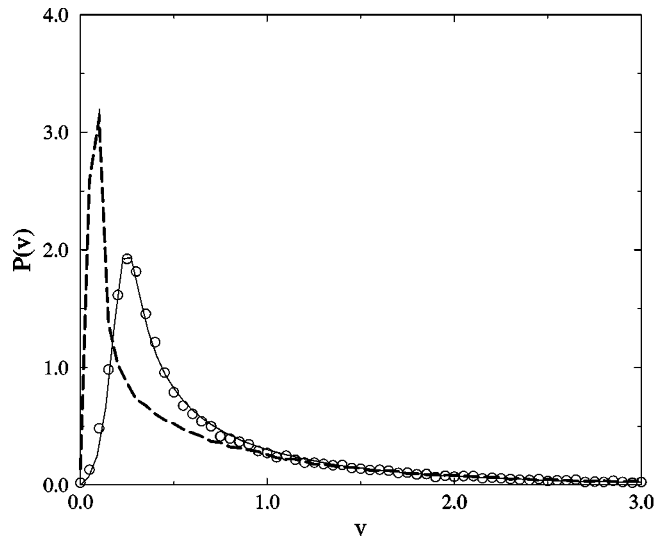

FIG. 1. Plot of $P(v,t)$ at $t=200$ for $P_0(v)=ve^{-v}$: simulation result (○) and numerical solution (—). The dashed line shows the simulation result for $P(v,t)$ at $t=16\,000$.

Equation (22) agrees with the previous nonrigorous argument.

A special consideration is required for the borderline case of $\mu=0$. For concreteness, consider the exponential intrinsic distribution, $P_0(v)=\exp(-v)$. Its double Laplace transform reads

$$\int_0^\infty dv\,\frac{e^{-v}}{s+v}=e^{s}E_1(s),$$

where $E_1(s)=\int_1^\infty dx x^{-1}e^{-xs}$ is the exponential integral. As a result, Eq. (19) becomes

$$\hat{u}(s)=-1+\frac{1}{se^{s}E_1(s)}. \quad (23)$$

Using the well-known asymptotics of the the exponential integral [9], $E_1(s)=-\ln s-\gamma+O(s)$, (where $\gamma\cong 0.5772$ is Euler's constant), we transform Eq. (23) into

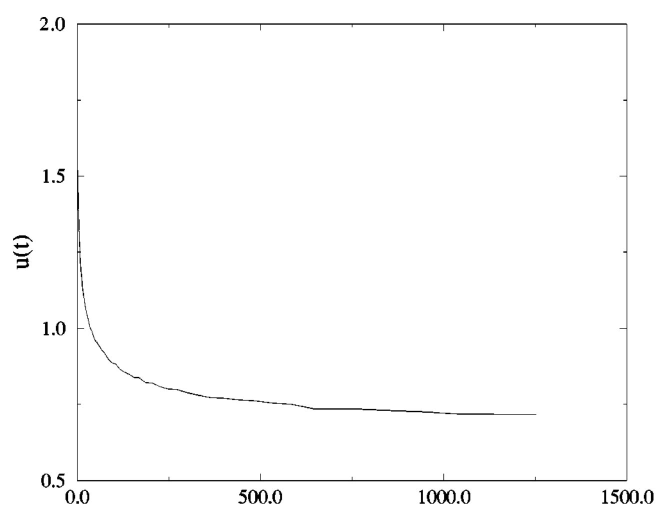

FIG. 2. Plot of $u(t)$ vs time $t$ for $P_0(v)=ve^{-v}$.

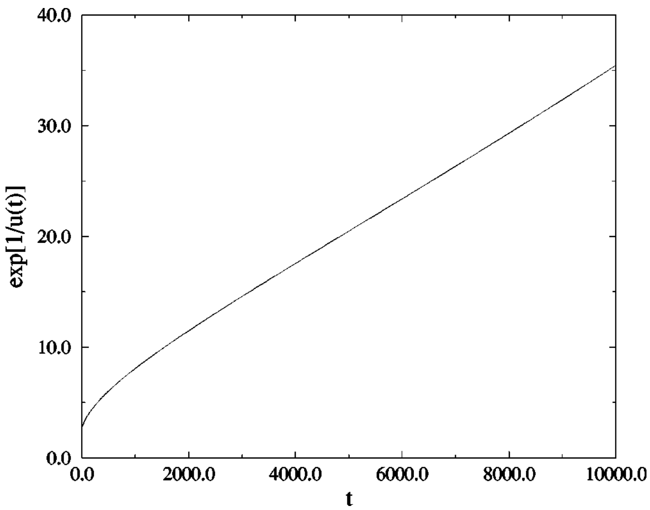


FIG. 3. Plot of $\exp[1/u(t)]$ vs time $t$ for $P_0(v)=e^{-v}$.

$$\hat{u}(s) = -\frac{1}{s(\ln s+\gamma)} + O\left(\frac{1}{\ln s}\right). \tag{24}$$

Performing the inverse Laplace transform we find that $u(t) \to (\ln t)^{-1}$ as $t\to\infty$. Thus, for the family of intrinsic velocity distribution with algebraic behavior (10) near the lower cutoff, our predictions for the long-time asymptotics of the average velocity $u(t)$ are

$$u(t) \sim \begin{cases} \text{const} & \text{for } \mu>0 \\ (\ln t)^{-1} & \text{for } \mu=0 \\ t^{\mu} & \text{for } -1<\mu<0. \end{cases} \tag{25}$$

To check the validity of asymptotic predictions and, more generally, to see if the mean-field theory is applicable at all, we perform molecular dynamics simulations and solve the Boltzmann equation (7) numerically. To sample distinct regimes predicted in Eq. (25) we consider $P_0(v) = v^{\mu}e^{-v}/\Gamma(\mu+1)$ with $\mu=1,0,-1/2$.

In molecular dynamics simulations, we place $N$ particles onto the ring of length $L=N$ so that the average density is equal to 1. Most of our simulations are performed for $N = 5\times 10^4$ particles, but we also simulated twice larger system and found no appreciable difference. Initially, particle velocities are randomly drawn from the distribution $P_0(v)$. The model is updated according to the collision-time-list algorithm suggested in Ref. [10]. To solve Eq. (7) numerically, we use Euler's time update with both uniform and nonuniform grids; in the latter case, we take $v_j=(j/j_{\max})^4 v_{\max}$ with $v_{\max}=15$ and $N_{\max}=500$. Integrals on the right-hand side of Eq. (7) are calculated using the trapezoid rule; time increment $\delta t=0.1$ was found to be suitable for all three $P_0(v)$. The results of molecular dynamics simulations and numerical solutions of the mean-field equation are virtually identical (see Fig. 1). Thus, the system remains well-stirred and no appreciable spatial correlations develop.

Figure 1 shows that for $P_0(v)=ve^{-v}$, the approach of $P(v,t)$ to the steady state is nonuniform in velocity. This is caused by the obvious fact that for any finite time, the velocity distribution $P(v,t)$ must still vanish at the lower cutoff as $P_0(v)$ does. The steady state is reached in the ''outer'' region $v \gg v_*(t)$, while in the ''inner,'' or boundary layer region the velocity distribution continues to evolve. The thickness $v_*(t)$ of the boundary layer is determined by a dominant-balance argument [9]. Since $v_*(t) \ll u_{\rm eq}$ as $t\to\infty$, we can consider Eq. (9) instead of the full Boltzmann equation (7). Balancing terms in Eq. (9) yields $t^{-1}P \sim v_*^{\mu+1}P$, which implies $v_* \sim t^{-1/(\mu+1)}$. Thus, the thickness of the boundary layer indeed shrinks with time but the boundary layer still exists *ad infinitum*. To determine a leading-order approximation to $P(v,t)$ as $t\to\infty$, one should separately solve for $P(v,t)$ in the outer and inner regions and then match the solutions. In the outer region, $P_{\rm out}(v,t)=P_{\rm eq}(v)$ and Eq. (7) simplifies to an integral equation. It is impossible to solve that equation in closed form, apart from the region $v \ll u_{\rm eq}$ where $P_{\rm eq}(v)$ is given by Eq. (11). In the inner region, the situation also considerably simplifies as the velocity distribution attains the scaling form $P_{\rm in}(v,t) = v_*^{\mu-1}\Phi(v/v_*)$. Matching inner and outer solutions yields $\Phi(\eta) \sim \eta^{\mu-1}$ as $\eta\to\infty$. Unfortunately, it is still impossible to solve for $\Phi(\eta)$.

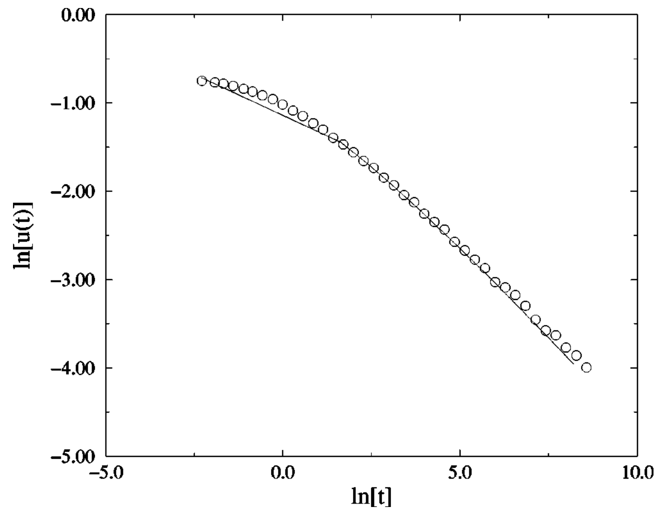


FIG. 4. Log-log plot of $u(t)$ vs time $t$ for $P_0(v) = (\pi v)^{-1/2}e^{-v}$: molecular dynamics results (○) and numerical solution (—).

Figures 2–4 plot the average velocity versus time for the intrinsic velocity distributions $P_0(v)=v^{\mu}e^{-v}/\Gamma(\mu+1)$ with $\mu=1,0,-1/2$, respectively. We find good agreement with the theoretical prediction of Eq. (25) when $\mu \geq 0$. For $\mu=-1/2$, the extrapolation of the local exponent $\alpha(t) \equiv d\ln[u(t)]/d\ln[t]$ to the $t\to\infty$ limit is in satisfactory agreement with $\alpha_{\rm theor}=\mu$.

In summary, we have shown that the fate of the system of passing particles is determined by the behavior of the intrinsic velocity distribution near its lower cutoff: If $P_0(v)$ vanishes in this limit, the system reaches a steady state; otherwise, the evolution continues forever. Comparison between solutions of the mean-field Boltzmann equation and results of molecular dynamics simulations suggests that the mean-field theory description is exact. It will be interesting to confirm this result rigorously.

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